Quantifying over Asynchronous Information Change

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Abstract

We propose a logic AAA for Arbitrary Asynchronous Announcements. In this logic, the sending and receiving of messages that are announcements are separated and represented by distinct modalities. Additionally, the logic has a modality that represents quantification over information change in the shape of sequences of sending and receiving events, called histories. We present a complete however infinitary axiomatisation, and various results for the logical semantics, wherein we consider both how the logic is different from asynchronous announcement logic AA and how the logic is different from arbitrary public announcement logic APAL. We also address the expressivity and we demonstrate the preservation of an extended fragment of positive formulas (wherein negations do not occur before epistemic modalities). Finally, we present work in progress on the logic AAM of Asynchronous Action Models and the logic AAAM of Arbitrary Asynchronous Action Models.

Keywords: Modal logic, dynamic epistemic logic, asynchrony, quantifying over information change

1 Introduction

We investigate what agents know and learn in distributed systems wherein the sending and receiving of messages are separated. Notions of asynchronous knowledge and common knowledge have been investigated in distributed computing in works such as [6,9,14,15,16] and in temporal epistemic logics in works such as [7,12,17,19]. Our take on such matters is from within so-called Dynamic Epistemic Logic (DEL) [18,5,22], a modal logic of knowledge and change of knowledge (or belief and change of belief), however not in its standard incarnation wherein message sending and receiving is synchronized and instantaneous, but in a recently investigated version by various researchers wherein these are separated [13,3,4]. These approaches are somewhat different from the asynchrony due to partial observation wherein histories (sequences of messages) of different length may have become indistinguishable for an agent, as in [8].

In [3] a logic AA is presented wherein messages that are announcements are still sent by an 'outside observer' or by the environment, but wherein they are individually received by the agents, unlike in public announcement logic [18] wherein all agents receive the announcement simultaneously (synchronised). The logic contains modalities for the announcement of φ, as in $[\varphi]\psi$. This has still as precondition that φ must be true when announced, but it does not have the effect the φ is received by any agent. For that, there are other modalities $[a]\psi$, for 'after the agent a has received the next announcement, ψ is true'. For example, we can now say that $[p][a]B_a p$: after p has been sent and agent a has received it, the agent knows/believes p. Therefore, in this logic AA we cannot obtain common knowledge that the announcement has been received, although we can still approach common knowledge by iterating announcements such as announcement p, all agents received p, announcement that everybody knows p, everybody received that, announcement that everybody knows that everybody knows p, etc. This is as in the concurrent common knowledge of [17]. In [3], a complete axiomatisation for such a logic is given, as well as special results on the class of $\mathcal{S}5$ models (where all relations are equivalence relations).

Subsequently, [4] investigates the wide spectrum of individual reception of messages up to synchronised reception by all agents of messages — and partial synchronisation for subgroups of all agents as well.

In the present work we generalize [3] in two ways: to the logic AA of asynchronous announcements we now add a quantifier $[!]\varphi$ for 'after any sequence of events, φ'. It is motivated by a similar quantifier in the logic APAL [1], that stands for 'after any/arbitrary announcement, φ'. Clearly, in the asynchronous setting we cannot have it merely quantifying over unreceived announcements, as this would not affect the beliefs or knowledge of agents. As the order of reception of announcements may vary greatly and may take place much later after an announcement, and possibly many subsequent announcements, have been sent, the natural form of quantification is therefore over arbitrary sequences of such sending and receiving events. We show that the resulting logic AAA has a complete axiomatisation, and varies in crucial respects from the motivating precedent APAL [1]. Such a logic of arbitrary asynchronous announcement may be, we hope, useful for diverse tasks such as: asynchronous epistemic planning, formalising epistemic protocols in distributed computing, and analysing the fine structure of interacting agents independently executing informative and other actions.

One particular further generalisation is also presented in some detail, namely the similar quantification over asynchronous non-public events (in the sense of events that are not known to be eventually received by all agents), such as an agent a privately receiving information on a proposition p, where an agent b also receives the information that a is privately receiving p although not necessarily simultaneous with a. From the works of Hales and collaborators [10,11] it is known that quantification over action models behaves much better than quantification over announcements: it is decidable, the quantifier can be eliminated from the language, and given $\langle ! \rangle \varphi$, for 'there is an action model after which φ', an action model can be synthesised that if executable always results in φ. We conjecture similar results for quantifying over asynchronous action models. In particular, asynchronous synthesis seems a highly desirable future goal.

In Section 2 we present Arbitrary Asynchronous Announcement logic AAA, for which we present various semantic results in Section 3. In Section 4 we address the expressivity, and in Section 5 the preservation (after history extension) of the fragment of *positive formulas*. Section 6 presents a complete infinitary axiomatisation. Finally, Section 7 adresses the generalisation of our results to a logic for quantification over asynchronous action models.

2 The logic AAA

Syntax. Let A be a finite set of epistemic agents and P a countable set of propositional variables. We consider the following language $\mathcal{L}$:

$$\varphi ::= p | \top | \neg\varphi | \varphi \vee \varphi | \hat{B}_a\varphi | \langle\varphi\rangle\varphi | \langle a\rangle\varphi | \langle ! \rangle\varphi,$$

where $p \in P, a \in A$.

We define duals $B_a\varphi = \neg\hat{B}_a\neg\varphi$, $[a]\varphi = \neg\langle a\rangle\neg\varphi$, $[\psi]\varphi = \neg\langle\psi\rangle\neg\varphi$, $[!]\varphi = \neg\langle ! \rangle\neg\varphi$.

Let $\mathcal{L}_{-!}$ be the fragment of this language without the $\langle ! \rangle$ modality.

Consider $A \cup \mathcal{L}$ as an alphabet, with agents and formulas as letters. Variables for words in this language are $\alpha, \beta, \ldots$, and ϵ denotes the empty word. Given a word α over $A \cup \mathcal{L}$, $|\alpha|$ is its length, $|\alpha|_a$ is the number of its a's (for each $a \in A$), $|\alpha|_!$ is the number of its formula occurrences, $\alpha\!\upharpoonright_!$ is the projection of α to $\mathcal{L}$, and $\alpha\!\upharpoonright_{!a}$ is the restriction of $\alpha\!\upharpoonright_!$ to the first $|\alpha|_a$ formulas. These notions have obvious inductive definitions.

For each such word, the formula $\langle\alpha\rangle\varphi$ represents an abbreviation of the sequence of announcement and reading modalities corresponding to the formulas and agents which appear in α, defined recursively as follows:

$$\langle\epsilon\rangle\varphi = \varphi;\ \langle\alpha.\psi\rangle\varphi = \langle\alpha\rangle\langle\psi\rangle\varphi;\ \langle\alpha.a\rangle\varphi = \langle\alpha\rangle\langle a\rangle\varphi,$$

where ϵ is the empty word. Every formula in $\mathcal{L}$ is thus of the form $\langle\alpha\rangle\varphi$ for some $\alpha \in (\mathcal{L} \cup A)^*$.

A *prefix* β of α, notation $\beta \subseteq \alpha$, is an initial sequence of α inductively defined as: $\alpha \subseteq \alpha$, and if $\beta \subseteq \alpha$, then for all $a \in A$ and $\psi \in \mathcal{L}$, $\beta \subseteq \alpha a$ and $\beta \subseteq \alpha\psi$.

For a sequence of announcements and readings to be executable, it is necessary that, whenever an agent is doing her n-th reading, there have been at least n formulas announced. Words satisfying this property will be called *histories*.

Histories. A word α in the language $A \cup \mathcal{L}$ is a *history* if for all prefixes $\beta \subseteq \alpha$ and for all $a \in A$, $|\beta|_! \geq |\beta|_a$.

We denote by $\mathcal{H}$ the set of histories. Obviously, if β is a prefix of a history α, then β is a history too.

View relation. Let α, β be histories and $a \in A$. We define: $\alpha \rhd_a \beta$ iff $|\beta|_a = |\alpha|_a$, $\beta\upharpoonright_{!a} = \alpha\upharpoonright_{!a}$ and $|\beta|_! = |\alpha|_a$. (Equivalently, iff $\alpha\upharpoonright_{!a} = \beta\upharpoonright_{!a} = \beta\upharpoonright_!$) The set $\mathsf{view}_a(\alpha) := \{\beta \mid \alpha \rhd_a \beta\}$ is the *view* of a given α. Informally, the view of agent a given history α consists of all the different ways in which a can receive the announcements in α. In other words, the view of a given α consists of the histories a considers possible (but without taking the meaning of the announcements in the history into account, which, as we will see, results in a further restriction). Note that $\mathsf{view}_a(\alpha)$ is a finite set.

Semantics. To define the semantics we will use the following well-founded preorder. First, we define $\deg_{\mathrm{B}} \varphi$, $\deg_! \varphi$ and $\|\varphi\|$ recursively: for $k = !, B$,

$\deg_k p = 0$	$\|p\| = 2$
$\deg_k \top = 0$	$\|\top\| = 1$
$\deg_k(\neg\varphi) = \deg_k \varphi$	$\|\neg\varphi\| = \|\varphi\| + 1$
$\deg_k(\varphi \wedge \psi) = \max\{\deg_k \varphi, \deg_k \psi\}$	$\|\varphi \wedge \psi\| = \|\varphi\| + \|\psi\|$
$\deg_k(\langle a\rangle\varphi) = \deg_k \varphi$	$\|\langle a\rangle\varphi\| = \|\varphi\| + 2$
$\deg_k(\langle\varphi\rangle\psi) = \deg_k \varphi + \deg_k \psi$	$\|\langle\varphi\rangle\psi\| = 2\|\varphi\| + \|\psi\|$
$\deg_{\mathrm{B}}(\hat{B}_a\varphi) = \deg_{\mathrm{B}} \varphi + 1$	$\|\hat{B}_a\varphi\| = \|\varphi\| + 1$
$\deg_!(\hat{B}_a\varphi) = \deg_! \varphi$	
$\deg_B(\langle!\rangle\varphi) = \deg_B \varphi$	$\|\langle!\rangle\varphi\| = \|\varphi\| + 1$
$\deg_!(\langle!\rangle\varphi) = \deg_! \varphi + 1$	

For a word α, we set $\deg_k \alpha := \sum\{\deg_k \psi : \psi \text{ occurs in } \alpha\}$ and

$$\|\epsilon\| = 0, \|\alpha.a\| = \|\alpha\| + 1, \|\alpha.\psi\| = \|\alpha\| + \|\psi\|.$$

Finally, for pairs (α, φ) we set: $\deg_k(\alpha, \varphi) = \deg_k \alpha + \deg_k \varphi$, and $\|(\alpha, \varphi)\| = \|\alpha\| + \|\varphi\|$, and we define a well-founded order $\ll$ as a lexicographical ordering on these quantities, i.e. $(\alpha, \varphi) \ll (\beta, \psi)$ iff

$$\begin{cases} \deg_!(\alpha,\varphi) < \deg_!(\beta,\psi), \text{ or} \\ \deg_!(\alpha,\varphi) = \deg_!(\beta,\psi) \,\&\, \deg_B(\alpha,\varphi) < \deg_B(\beta,\psi), \text{ or} \\ \deg_!(\alpha,\varphi) = \deg_!(\beta,\psi) \,\&\, \deg_B(\alpha,\varphi) = \deg_B(\beta,\psi) \,\&\, \|(\alpha,\varphi)\| < \|(\beta,\psi)\|. \end{cases}$$

We interpret formulas on models (W, R, V), where $R : A \to \mathcal{P}(W^2)$, with respect to pairs (w, α) where $w \in W$ an $\alpha \in \mathcal{H}$. We define the relations "w agrees with α" ($w \bowtie \alpha$) and "(w, α) satisfies φ" ($w, \alpha \models \varphi$) by $\ll$-induction as it appears in Table 1. A formula φ is ϵ-valid, notation $\models^\epsilon \varphi$, iff for all models (W, R, V) and for all $s \in W$, $s, \epsilon \models \varphi$. A formula φ is $*$-valid, notation $\models^* \varphi$, iff for all models (W, R, V), for all $s \in W$ and for all histories α, $s, \epsilon \models [\alpha]\varphi$.

Note that the $\langle!\rangle$ modality only quantifies over words wherein $\langle!\rangle$ does not occur. This is to avoid a circular definition. The dual of $\langle!\rangle$ is read as follows:

$w \bowtie \epsilon$	always;
$w \bowtie \alpha.\varphi$	iff $w \bowtie \alpha$ and $w, \alpha \models \varphi$;
$w \bowtie \alpha.a$	iff $w \bowtie \alpha$;
$w, \alpha \models p$	iff $w \in V(p)$;
$w, \alpha \models \top$	always;
$w, \alpha \models \neg\varphi$	iff $w, \alpha \not\models \varphi$;
$w, \alpha \models \varphi_1 \wedge \varphi_2$	iff $w, \alpha \models \varphi_i$, $i = 1, 2$;
$w, \alpha \models \langle a \rangle \varphi$	iff $\lvert\alpha\rvert_a < \lvert\alpha\rvert_!$ and $w, \alpha.a \models \varphi$;
$w, \alpha \models \langle \psi \rangle \varphi$	iff $w, \alpha \models \psi$ and $w, \alpha.\psi \models \varphi$;
$w, \alpha \models \hat{B}_a \varphi$	iff $t, \beta \models \varphi$ for some $(t, \beta) \in W \times \mathcal{H}$ such that $R_a wt, \alpha \triangleright_a \beta, t \bowtie \beta$
$w, \alpha \models \langle ! \rangle \varphi$	iff $w, \alpha \models \langle \beta \rangle \varphi$ for some $\beta \in (\mathcal{L}_{-!} \cup A)^*$.

Table 1
Semantics of AAA

$w, \alpha \models [!]\varphi$ if and only if, for every possible sequence $\beta \in (\mathcal{L}_{-!} \cup A)^*$, it is the case that $w, \alpha \models [\beta]\varphi$.

Note moreover that the relation $\triangleright_a$ is not reflexive (it is however postreflexive, in the sense that $\alpha \triangleright_a \beta$ implies $\beta \triangleright_a \beta$). For this reason, it is not the case that $w, \alpha \models B_a\varphi$ implies $w, \alpha \models \varphi$. Our modality is not factual and this is the reason we favour a doxastic interpretation of it over an epistemic one.

We make the assumption that an agent forms her beliefs based on announcements she has so far received, ignoring possible future announcements (indeed, note that $B_a[a]\bot$ always true: an agent never believes there are unread announcements).

The following lemma, whose proof is straightforward, will be useful:

Lemma 2.1 *Given a model (W, R, V), $w \in W$, $\varphi \in \mathcal{L}$, $\alpha \in \mathcal{H}$ such that $w \bowtie \alpha$, and $\beta \in (\mathcal{L} \cup A)^*$, the following are equivalent:*

i. $w, \alpha \models \langle \beta \rangle \varphi$;

ii. The concatenation $\alpha.\beta$ is a history, $w \bowtie \alpha.\beta$, and $w, \alpha.\beta \models \varphi$.

3 Semantic results for the logic AAA

Bisimulation. The notion of bisimulation in this framework is, perhaps surprisingly, the usual notion of bisimulation between Kripke models: given (W, R, V) and (W', R', V'), a *bisimulation* is a relation $Z \subseteq W \times W'$ such that, if wZw':

i. $w \in V(p)$ iff $w' \in V'(p)$;

ii. if $R_a wv$, then there exists $v' \in W'$ such that $R'_a w'v'$ and vZv';

iii. if $R'_a w'v'$, then there exists $v \in W$ such that $R_a wv$ and vZv'.

As one might expect, we have the following:

Proposition 3.1 *Let Z be a bisimulation such that wZw', and let $(\alpha, \varphi) \in \mathcal{H} \times \mathcal{L}$. We have:*

$$w, \epsilon \models \langle\alpha\rangle\varphi \text{ iff } w', \epsilon \models \langle\alpha\rangle\varphi.$$

Proof. See Appendix. □

Under certain constraints, if two states satisfy the same formulas, they are bisimilar. Indeed:

Proposition 3.2 *Let (W, R, V) and (W', R', V') be two models such that $R_a[w]$ and $R'_a[w']$ are finite sets for all $w \in W$, $w' \in W'$. Set wZw' iff, for all $(\alpha, \varphi) \in \mathcal{H} \times \mathcal{L}$, $w, \epsilon \models \langle\alpha\rangle\varphi$ iff $w', \epsilon \models \langle\alpha\rangle\varphi$. Then Z is a bisimulation.*

Proof. See Appendix. □

Properties of belief. As discussed above, while $B_a\varphi \to \varphi$ is ϵ-valid, (as long as the relation R_a is reflexive) it is not $*$-valid. Other properties of our doxastic modality, however, are $*$-valid. Let $\mathsf{S5}$ denote the class of models where the relations R_a are equivalence relations. We have:

Proposition 3.3 *Let $\varphi \in \mathcal{L}$. Then:*

i. $\mathcal{S}5 \models^* B_a\varphi \to \neg B_a\neg\varphi$

ii. $\mathcal{S}5 \models^* B_a\varphi \to B_aB_a\varphi$

iii. $\mathcal{S}5 \models^* \neg B_a\varphi \to B_a\neg B_a\varphi$

Proof. See Appendix. □

Belief before and after update. If an agent will believe φ after a certain sequence of events then the agent believes that there is a sequence of events after which φ holds, but not the other way around. Indeed:

Proposition 3.4 $\models^\epsilon \langle!\rangle\hat{B}_a\varphi \to \hat{B}_a\langle!\rangle\varphi$, *whereas* $\not\models^\epsilon \hat{B}_a\langle!\rangle\varphi \to \langle!\rangle\hat{B}_a\varphi$.

Proof. See Appendix. □

Church-Rosser and McKinsey Let us see that neither of the formulas

(CR) $\langle!\rangle[!]\varphi \to [!]\langle!\rangle\varphi$ (McK) $[!]\langle!\rangle \to \langle!\rangle[!]\varphi$

are sound. It is known from APAL that these properties are valid for arbitrary announcement on the class of $\mathcal{S}5$ models (where all accessibility relations are equivalence relations) [1]. As we consider arbitrary relations, this is not unexpected. We address the case $\mathcal{S}5$ a the end of this paragraph.

First let us see (McK) is not sound. Let $\varphi = [a]\bot$. Then φ will be satisfied at a pair w, α if and only if $|\alpha|_a = |\alpha|_!$. For any history β it is the case that $|\beta|_a \leq |\beta|_!$: let $\delta_\beta = a...a$ be the concatenation of $|\beta|_! - |\beta|_a$ times the letter a. Then, for every β there exists a word δ_β such that $w, \epsilon \models [\beta]\langle\delta_\beta\rangle[a]\bot$. However, $\langle!\rangle[!][a]\bot$ is never satisfied: indeed, for any history β, let δ_β be a concatenation of the formula $\top$ enough times so that $|\beta\delta_\beta|_! > |\beta|_a$. Then we have $w, \epsilon \not\models \langle\beta\rangle[\delta_\beta][a]\bot$.

Let us now see a counterexample for (CR). Consider the following one-agent model [1] :

[1] We thank Louwe Kuijer for this counterexample

Let $W = \{w_1, w_2, w_3\}$, $R_a = \{(w_1, w_2), (w_2, w_2), (w_2, w_3)\}$ and $V(p) = \{w_1, w_2\}$. We have that $w_1, \epsilon \models \langle ! \rangle [!] \hat{B}_a \top$. Indeed, consider the history $p.a.[a]\bot.a$. We can easily prove the following by induction on φ:

If β is a history having $p.a.[a]\bot.a$ as a prefix, then for all φ, $w_1, \beta \models \varphi$ iff $w_2, \beta \models \varphi$.

In particular, any β having $p.a.[a]\bot.a$ as a prefix will be executable at w_1 iff it is executable at w_2. Now, take any sequence γ such that $p.a.[a]\bot.a.\gamma$ is executable at w_1. There exists a β such that $p.a.[a]\bot.a\gamma \triangleright_a \beta$ and β is executable at w_1. Note that β is necessarily of the form $\beta = p.a.[a]\bot.a.\gamma'$ for some γ'. But this means, by the previous remark, that β is executable at w_2, and thus $w_1, p.a.[a]\bot.a.\gamma \models \hat{B}_a \top$, which means $w_1, \epsilon \models \langle p.a.[a]\bot.a \rangle [!] \hat{B}_a \top$ and thus $w_1, \epsilon \models \langle ! \rangle [!] \hat{B}_a \top$.

However, $w_1, \epsilon \not\models [!] \langle ! \rangle \hat{B}_a \top$: indeed, consider the sequence $B_a p.a$. It is never the case that $w_1, B_a p.a \models \langle \beta \rangle \hat{B}_a \top$ for any announcement, given that, whenever $B_a p.a.\beta \triangleright_a \gamma$, γ has $B_a p$ as its first formula, and therefore γ cannot be compatible with w_2, since $w_2, \epsilon \not\models B_a p$.

Also in APAL (CR) is not sound in general (this can be seen via a similar counterexample), but, as said, only with equivalence relations. Whether CR is sound on AAA on the class of $\mathcal{S}5$ models is an open question.

4 Expressivity of AAA

We assume the usual terminology to compare the expressivity of logics or logical languages with respect to a semantics and a class of models. Given two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ interpreted over the same class $\mathcal{C}$ of models, we say that $\mathcal{L}_1$ is at least as expressive as $\mathcal{L}_2$ with respect to $\mathcal{C}$ iff for all formulas $\varphi_2 \in \mathcal{L}_2$, there exists a formula $\varphi_1 \in \mathcal{L}_1$ such that for all models $\mathfrak{M} \in \mathcal{C}$, $\mathfrak{M} \models \varphi_1$ iff $\mathfrak{M} \models \varphi_2$.

If $\mathcal{L}_1$ is at least as expressive as $\mathcal{L}_2$ and $\mathcal{L}_2$ is at least as expressive as $\mathcal{L}_1$ then $\mathcal{L}_1$ and $\mathcal{L}_2$ are *as expressive*. If $\mathcal{L}_1$ is at least as expressive as $\mathcal{L}_2$ and $\mathcal{L}_2$ is not at least as expressive as $\mathcal{L}_1$ then $\mathcal{L}_1$ is *more expressive than* $\mathcal{L}_2$. In this section we show that the language of AAA is more expressive than that of AA, by showing that there is a formula $\varphi \in \mathcal{L}$ to which no formula $\varphi' \in \mathcal{L}_{-!}$ is equivalent. This is shown somewhat similarly to proving that APAL is more expressive than PAL [1, Proposition 3.13]. [2]

Proposition 4.1 *$\mathcal{L}$ is more expressive than $\mathcal{L}_{-!}$ for multiple agents, for the class $\mathcal{S}5$ of models wherein each R_a is an equivalence relation.*

Proof. Suppose that AAA is as expressive as AA in $\mathcal{S}5$ for multiple agents. Consider the formula $\langle ! \rangle (B_a p \wedge B_a \neg B_b p)$. Then there must be a formula $\varphi \in \mathcal{L}_{-!}$

[2] However, with differences that may be considered of interest. In the APAL proof the property used to demonstrate larger expressivity is $\langle ! \rangle (B_a p \wedge \neg B_b B_a p)$. This property uses that in APAL an announcement results in a growth of common knowledge, it uses the synchronous character of PAL announcements. We use another property, $\langle ! \rangle (B_a p \wedge B_a \neg B_b p)$, and on a slightly different model.

that is equivalent to $\langle ! \rangle (B_a p \wedge B_a \neg B_b p)$. Some propositional variable q will not occur in φ. Now consider $\mathcal{S}5$ models M and M' as below (indistinguishable states are linked, and we assume transitivity of access). Of course, the states in M also need a value for atom q, but this is irrelevant for the proof and therefore not depicted (for example, we can assume q to be false in both).

$$M: \quad \begin{tikzcd} t(\neg p) \arrow[r, dash, "ab"] & s(p) \end{tikzcd} \qquad\qquad M': \quad \begin{tikzcd} t'(\neg p q) \arrow[r, dash, "ab"] \arrow[d, dash, "b"'] & s'(pq) \arrow[d, dash, "b"] \\ u'(\neg p \neg q) \arrow[r, dash, "ab"] & v'(p \neg q) \end{tikzcd}$$

We note that (M, s) is bisimilar to (M', s') if we restrict the clause (i) (for corresponding valuations) to the variable p only. If we now consider formulas $\varphi \in \mathcal{L}_{-!}$ and histories $\alpha \in (\mathcal{L}_{-!} \cup A)^*$ that do not contain the variable q, it can be easily shown by induction on (α, φ) that $s \bowtie \alpha$ iff $s' \bowtie \alpha$ and $s, \alpha \models \varphi$ in M if and only if $s', \alpha \models \varphi$ in M'. As a consequence, for any $\varphi \in \mathcal{L}_{-!}$, we have that $s, \epsilon \models \varphi$ iff $s', \epsilon \models \varphi$.

However, this is not the case in the full language $\mathcal{L}$. We then have that $s, \epsilon \not\models \langle ! \rangle (B_a p \wedge B_a \neg B_b p)$ in M, whereas $s', \epsilon \models \langle ! \rangle (B_a p \wedge B_a \neg B_b p)$ in M'. The former is because in M, for any history α only executable in s, for any announcement in α received by a, a considers it possible that b also received this announcement and thus believes p. The latter is because in M' it holds that $s', (p \vee \neg q).a.b \models B_a p \wedge B_a \neg B_b p$.

This is a contradiction. □

It seems likely, although we did not prove this, that on class $\mathcal{S}5$ for a single agent the $\langle ! \rangle$ modality is definable in AA, such that AAA is then as expressive as AA. However, without any frame properties single-agent AAA is more expressive than AA, again shown similarly to the previous proposition and [1, Prop. 3.14]

Proposition 4.2 *$\mathcal{L}$ is more expressive than $\mathcal{L}_{-!}$ for a single agent.*

Proof. See Appendix. □

A logic is called *compact* if a set of formulas is satisfiable whenever any finite subset is satisfiable.

Proposition 4.3 *The logic* AAA *is not compact.*

Proof. Using the expressivity results, this can be shown by considering the set of formulas

$$\{\langle ! \rangle (B_a p \wedge B_a \neg B_b p)\} \cup \{\neg \langle \beta \rangle (B_a p \wedge B_a \neg B_b p) : \beta \in (\mathcal{L}_{-!} \cup A)^*\}.$$

This set is not satisfiable, but any finite subset is satisfiable, where we use that some variable q must necessarily not occur in such a subset. We then consider M, M' as above. The q-less finite subset will be satisfied at s'. □

5 Positive formulas

In modal logic, the fragment of the language where negations do not bind epistemic modalities is known as the *positive* fragment [20,21,1]. It corresponds to the universal fragment in first-order logic. It has the property that it preserves truth under submodels. In AAA, preservation under submodels is formalised by preservation after history extension. A formula $\varphi \in \mathcal{L}$ is *preserved* iff $\models^* \varphi \rightarrow [!]\varphi$. We wish therefore to identify a fragment of the language $\mathcal{L}$ that guarantees preservation.

For AA, it is shown in [3, Prop. 44] that the fragment $\varphi ::= p \mid \neg p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid B_a\varphi$, that corresponds in a very direct way to the universal fragment, is preserved.

For AAA we wish to expand that frontier, in the direction earlier taken in [21] for synchronous announcements, where a further inductive clause $[\neg\varphi]\varphi$ is added, which is further expanded in [1] with an inductive clause $[!]\varphi$ (where $[!]$ is the APAL quantifier over announcements). We will only define a fairly minimal extension and subsequently present some of the difficulties in obtaining a result analogous to those in [21,1], and what the desirable final goal seems to be.

The proof uses a lemma that we therefore present first. Let preorder $\preceq$ on histories be defined as follows: $\alpha \preceq \beta$ iff $\alpha{\restriction_!} \subseteq \beta{\restriction_!}$, for all $a \in A$ $|\alpha|_a \leq |\beta|_a$, and for any state s in any model $s \bowtie \beta$ implies $s \bowtie \alpha$. Note that $\alpha \subseteq \beta$ implies $\alpha \preceq \beta$, but not vice versa.

Lemma 5.1 ([3, Lemma 42]) *Let histories α, β and $a \in A$ be given. Suppose $\alpha \preceq \beta$ and $\beta \rhd_a \delta$. Then there is a history γ such that $\gamma \preceq \delta$ and $\alpha \rhd_a \gamma$.*

Consider the following *positive formulas* $\mathcal{L}_+$:

$$\varphi ::= p \mid \neg p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid B_a\varphi \mid [!]\varphi.$$

We show that positive formulas are preserved.

Proposition 5.2 (Positive implies preserved)
Let $\varphi \in \mathcal{L}_+$. Then $\models_* \varphi \rightarrow [!]\varphi$.

Proof. We need to prove the following proposition:

> Let $\varphi \in \mathcal{L}_+$. For all models $M = (W, R, V)$ and $s \in W$, and for all histories α: $s, \epsilon \models [\alpha](\varphi \rightarrow [!]\varphi)$.

This is equivalent to

> Let $\varphi \in \mathcal{L}_+$. For all models $M = (W, R, V)$ and $s \in W$, and for all histories α, β such that $\alpha \subseteq \beta$: $s, \epsilon \models [\alpha]\varphi$ implies $s, \epsilon \models [\beta]\varphi$.

A standard inductive proof on the structure of φ fails because in the case $B_a\varphi$ we would need that if $\alpha \subseteq \beta$ and $\beta \rhd_a \delta$, then there is a γ with $\gamma \subseteq \delta$ and $\alpha \rhd_a \gamma$. Such a γ may not exist, namely if many yet unread announcements in δ precede the a in δ that corresponds to the last a in α. However, we can then still find a γ such that $\gamma \preceq \delta$. Therefore, it suffices to show:

Lemma 5.3 *Let $\varphi \in \mathcal{L}_+$. For all models $M = (W, R, V)$ and $s \in W$, and for all histories α, β such that $\alpha \preceq \beta$: $s, \epsilon \models [\alpha]\varphi$ implies $s, \epsilon \models [\beta]\varphi$.*

A proof of this Lemma can be found in the Appendix. □

With this definition of preservation we cannot include an obvious clause for announcement into the inductive definition of positive formulas, where the obvious analogue of the $[\neg\varphi]\psi$ from [21] would be $[\neg\varphi.A]\psi$ (and where A represents an arbitrary permutation of all agents in A). [3] For example, consider a model M for one agent a and two variables p, q consisting of four worlds for the four valuations of p and q, and such that these are all indistinguishable for a. Let w be the world where p and q are true. We now have that: $w, \epsilon \models [q.a]B_a q$ whereas $w, p \not\models [q.a]B_a q$, because the a in history $q.a$ reads announcement q in the first case whereas it reads announcement p in the second case. As long as agent a has not received announcement q, she remains uncertain about the value of q.

According to some such clause $[\neg\varphi.A]\psi$ (and other clauses for atoms, their negation, and belief), $[q.a]B_a q$ should be a positive formula. But then we no longer have $\models^* \varphi \to [!]\varphi$.

Beyond having $[\neg\varphi.A]\psi$, that would also make $[p.a.p.a]B_a p$ positive (which is the same as $[p.a][p.a]B_a p$), should we then not want $[p.p.a.a]B_a p$ to be positive, where both announcements are only read after they have been announced?

It seems that the definition of preservation as $\models^* \varphi \to [!]\varphi$, where $[!]$ quantifies over words instead of over histories, and where histories may contain unread announcements, effectively rules out the inclusion of announcements and read modalities in a positive fragment. It may even be that the positive fragment as defined syntactically characterizes the preserved formulas (with respect to $*$-validity), analogous to van Benthem's result for the (usual) positive fragment [20]. This we do not know yet. Alternatively, a definition of preserved with respect to ϵ-validity (so, ϵ-preserved) might well be $\models^\epsilon \varphi \to [!](\bigwedge_{a \in A}[a]\bot \to \varphi)$. This would allow a more liberal fragment of positive formulas including the above examples. We also wish to investigate that in future research, and where again the ultimate goal is a syntactic characterisation of ϵ-preservation.

Before moving on, let us point out another property of the positive fragment: when the believed formula φ is positive, and the accessibility relation reflexive, belief becomes factive.

Proposition 5.4 *Let $\varphi \in \mathcal{L}_+$. For any model (W, R, V) such that R_a is reflexive, for all $s \in W$ and α such that $s \bowtie \alpha$, we have $s, \alpha \models B_a\varphi \to \varphi$. As a consequence, $\mathcal{S}5 \models^* B_a\varphi \to \varphi$.*

Proof. Suppose $s, \alpha \models B_a\varphi$. Consider $\beta = \alpha.\varphi.a^k$, as constructed in the proof of Prop. 3.3. We have $R_a ss$, $\alpha \triangleright_a \beta$, and $s \bowtie \beta$, and thus $s, \beta \models \varphi$. Moreover,

[3] The possibly strange form of this clause wherein a negation appears has to do with the semantics of public announcement. In PAL, $M, w \models [\neg\varphi]\psi$ iff ($M, w \models \neg\varphi$ implies $M', w \models \psi$) iff ($M, w \models \varphi$ or $M', w \models \psi$), where M' is the model restriction to the states where φ is true. In the disjunctive description, the negation has disappeared.

since $\delta.\varphi \subseteq \alpha$ and $|\beta|_a = |\alpha|_a$, we have $\beta \preceq \alpha$. By Lemma 5.3, this entails $s, \alpha \models \varphi$. □

6 Axiomatisation of AAA

The axiomatisation of AAA and its completeness proof is based on the axiomatisation of AA [3] and on that of APAL [1] and its completeness uses the method pioneered in [2].

We will say that a formula $\varphi \in \mathcal{L}$ is *ϵ-valid* if, for every model (W, R, V) and every $w \in W$, it is the case that $w, \epsilon \models \varphi$, and φ is *$*$-valid* if, for every model (W, R, V) and $w \in W$, and for every history α such that $w \bowtie \alpha$, it is the case that $w, \alpha \models \varphi$. In the the present section we provide a complete axiomatization of the logic of ϵ-validities.

Given a symbol # we define a set AF of *admissible forms* as follows:

$$L ::= \# | B_a L | \varphi \to L | \langle \alpha \rangle L,$$

where $\varphi \in \mathcal{L}$, $a \in A$, $\alpha \in \mathcal{H}$, $L \in AF$. Given $L \in AF$ and $\varphi \in \mathcal{L}$, the formula $L(\varphi)$ is the result of substituting the unique occurrence of # in L by φ.

The following holds:

Lemma 6.1 *Let L be an admissible form. For all $M \in AF$ and for all modal formulas φ, ψ, if $L([!]\varphi) = M([!]\psi)$ then $L = M$ and $\varphi = \psi$.*

Proof. By induction on L. □

The logic AAA consists of the following axioms and rules, for $\alpha \in \mathcal{H}$, $p \in P$, $a \in A$, $L(\#) \in AF$

(MP)	If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$
(Nec$_B$)	If $\vdash \varphi$, then $\vdash B_a \varphi$
(K$_B$)	$B_a(\varphi \to \psi) \to (B_a \varphi \to B_a \psi)$
($R_{\top 1}$)	$\langle \alpha.a \rangle \top \leftrightarrow \langle \alpha \rangle \top$ if $\lvert\alpha\rvert_a < \lvert\alpha\rvert_!$
($R_{\top 2}$)	$\langle \alpha.a \rangle \top \leftrightarrow \bot$ otherwise;
($R_{\top 3}$)	$\langle \alpha.\varphi \rangle \top \leftrightarrow \langle \alpha \rangle \varphi$;
(R_p)	$\langle \alpha \rangle p \leftrightarrow (\langle \alpha \rangle \top \wedge p)$;
($R_\neg$)	$\langle \alpha \rangle \neg \varphi \leftrightarrow (\langle \alpha \rangle \top \wedge \neg \langle \alpha \rangle \varphi)$;
($R_\vee$)	$\langle \alpha \rangle (\varphi \vee \psi) \leftrightarrow (\langle \alpha \rangle \varphi \vee \langle \alpha \rangle \psi)$;
(R_B)	$\langle \alpha \rangle \hat{B}_a \varphi \leftrightarrow (\langle \alpha \rangle \top \wedge \bigvee_{\alpha \triangleright_a \beta} \hat{B}_a \langle \beta \rangle \varphi)$;
([!]-elim)	$L([!]\varphi) \to L([\beta]\varphi)$ (where $\beta \in (\mathcal{L}_{-!} \cup A)^*$);
([!]-int$^\omega$)	If $\vdash L([\beta]\varphi)$ for all $\beta \in (\mathcal{L}_{-!} \cup A)^*$, then $\vdash L([!]\varphi)$

Remark 6.2 *If we remove the last two lines of the above table we obtain the logic* AA*, defined in [3] for the language $\mathcal{L}_{-!}$.*

Completeness proof. A *theory* is a set of formulas T such that:

i. AAA $\subseteq T$;
ii. T is closed under Modus Ponens: if $\varphi, \varphi \to \psi \in T$, then $\psi \in T$;
iii. T is closed under the following rule:
If $L([\beta]\varphi) \in T$ for all $\beta \in (\mathcal{L}_{-!} \cup A)^*$, then $L([!]\varphi) \in T$.

A theory is *consistent* if $\bot \notin T$. Note that AAA is the least consistent theory, and $\mathcal{L}$ is the only inconsistent theory.

A consistent theory is *maximal* if no proper superset of T is a consistent theory.

The following holds:

Lemma 6.3 *Given a theory T, a formula ψ, and an agent $a \in A$, the sets $T_{B_a} = \{\varphi : B_a\varphi \in T\}$ and $T_\psi = \{\varphi : \psi \to \varphi \in T\}$ are also theories.*

Moreover, $T \subseteq T_\psi$, $\psi \in T_\psi$ and, if $\neg\psi \notin T$, then T_ψ is consistent.

Proof. See Appendix. □

We also have:

Proposition 6.4 (Lindenbaum's Lemma) *A consistent theory can be extended to a maximal consistent theory.*

Proof. See Appendix. □

Now we define a relation between maximal consistent theories as: TR_aS iff, for all φ, $B_a\varphi \in T$ implies $\varphi \in S$ (equivalently, iff $T_{B_a} \subseteq S$).

Proposition 6.5 (Diamond Lemma) *Suppose $\hat{B}_a\varphi \in T$. Then there exists a maximal consistent theory S such that TR_aS and $\varphi \in S$.*

Proof. Consider the theory $(T_{B_a})_\varphi$. First, note that T_{B_a} is a consistent theory, because $\vdash \hat{B}_a\varphi \to \neg B_a\bot$, so $B_a\bot \notin T$ and thus $\bot \notin T_{B_a}$. Moreover, $B_a\neg\varphi \notin T$, thus $\neg\varphi \notin T_{B_a}$. By Lemma 6.3, we thus have that $T_{B_a} \subseteq (T_{B_a})_\varphi$, $\varphi \in (T_{B_a})_\varphi$ and $(T_{B_a})_\varphi$ is consistent. It then suffices to extend $(T_{B_a})_\varphi$ by Lindenbaum's lemma to the desired successor. □

Now we can defined our canonical model: let W be the family of maximal consistent theories, let R_a be defined as above and let $V(p) = \{T \in W : p \in T\}$. We have:

Proposition 6.6 (Truth Lemma) *For any history α and formula φ, we have: $T, \epsilon \models \langle\alpha\rangle\varphi$ iff $\langle\alpha\rangle\varphi \in T$.*

Proof. See Appendix. □

We will say that a formula φ is *consistent* if $\nvdash \neg\varphi$ and that a set of formulas Γ is *consistent* if it can be extended to a consistent theory. Note that φ is consistent if and only if the singleton set $\{\varphi\}$ is consistent (for if $\neg\varphi \notin \mathsf{AAA}$, we can extend $\{\varphi\}$ to the consistent theory AAA_φ).

We have:

Theorem 6.7 AAA *is strongly complete with respect to Kripke models.*

Proof. Let Γ be a consistent set of formulas. Then there exists a consistent theory $T_0 \supseteq \Gamma$ and, by Lindenbaum's lemma, a maximal consistent theory $T \supseteq T_0$. We construct the canonical model as above and we have that $T, \epsilon \models \varphi$ for all $\varphi \in \Gamma$. □

7 Asynchronous Action Models

In this final section we shortly present two logics for asynchronous reception of partially observed actions, including quantification over such actions. The reason to present these logics is that they contrast in, we think, interesting ways with the logic AA and with the logic AAA, the main subject of this paper.

7.1 Asynchronous Action Model Logic

Action model logic was proposed by Baltag, Moss and Solecki in [5]. An *action model* is like a relational model but the elements of the domain are called *actions* instead of states, and instead of a valuation a *precondition* is assigned to each domain element. A public announcement corresponds to a singleton action model where the precondition is the formula of the announcement. Under synchronous conditions, executing an action model into a Kripke model means constructing what is known as the *restricted modal product.* This product encodes the new state of information, after action execution. Under asynchronous conditions we do not construct the product model but calculate the belief consequences of actions from the histories, just as for the particular singleton action model that is the public announcement we do not construct model restrictions in AA but instead use the history.

The nature of an asynchronous non-public action is that it is partially observed by the agents, just as in action model logic, but that it is unclear when the different agents partially observe the action, just as in AA. An example of an asynchronous partially observed action when two agents Anne and Bill, who are both ignorant about p, are informed that Anne will receive the truth about some proposition p but not Bill. Suppose that Anne is going to receive the information that p (is true). By the time Bill learns that Anne will be informed in this way, he considers it possible that Anne has already been informed, in which case she now believes p or believes $\neg p$, but he also considers it possible that she has not yet been informed and thus remains igorant about p. Dually, by the time Anne learns that p but Bill has not yet learnt that Anne will be informed about p, Bill incorrectly believes that Anne is ignorant about p.

Action model Formally, an *action model* $\mathcal{E} = (E, S, \mathsf{pre})$ consists of a *domain* E of *actions* $e, f, \dots$, an *accessibility function* $S : A \to \mathcal{P}(E^2)$, where each S_a is an accessibility relation, and a *precondition function* $\mathsf{pre} : E \to \mathcal{L}$, where $\mathcal{L}$ is a logical language. A pointed action model is a pair $(\mathcal{E}, e)$ where $e \in E$, for which we write $\mathcal{E}_e$. We abuse the language and also call a pointed action model an *action.*

Syntax Similarly to AA we can conceive a modal logical language with $\langle \mathcal{E}_e \rangle \varphi$ as an inductive language construct, for action models $\mathcal{E}$ with *finite domains.* The class of finite pointed action models is called $\mathcal{AM}$.

Histories are words in $(\mathcal{AM} \cup A)^*$. Much like in AA, we will use $\alpha\!\upharpoonright_!$ to refer to the projection of α to $\mathcal{AM}$ and use $\alpha \upharpoonright_{!a}$, $|\alpha|_!$, $|\alpha|_a$ as usual.

View relation The definition of the $\triangleright_a$ relation in this setting incorporates the partial observablity of action models: given $\alpha \triangleright_a \beta$, we demand that the *action models* appearing in α and β are the same. However, for agent a the *actions* in α (points of these action models) may be different from the actions in β. That is, $\alpha \rhd_a \beta$ iff $|\alpha|_a = |\beta|_a = |\beta|_!$, and for all $i \leq |\alpha|_a$, if $\mathcal{E}_e$ is the i-th action of α and $\mathcal{F}_e$ is the ith action of β, then $\mathcal{E} = \mathcal{F}$ and $S_a ef$. This relation $\rhd_a$ is post-reflexive, transitive and post-symmetric if we are dealing with $\mathcal{S}5$ action models (wherein all accessibility relations S_a are equivalence relations).

Semantics We define an executibility relation $\bowtie$ as follows:

- $w \bowtie \epsilon$,
- $w \bowtie \alpha.a$ iff $w \bowtie \alpha$,
- $w \bowtie \alpha.\mathcal{E}_e$ iff $w \bowtie \alpha$ and $w, \alpha \models \mathsf{pre}(e)$.

With this, the semantics for belief and action model execution are what one might expect, namely:

$w, \alpha \models \langle \mathcal{E}_e \rangle \varphi$ iff $w, \alpha \models \mathsf{pre}(e)$ and $w, \alpha.\mathcal{E}_e \models \varphi$.
$w, \alpha \models \hat{B}_a \varphi$ iff $t, \beta \models \varphi$ for some (t, β) such that $t \bowtie \beta, R_a wt,$ and $\alpha \rhd_a \beta$.

We call this *Asynchronous Action Model Logic* AAM.

Reduction axioms and axiomatisation We recall that the axiomatisation AAA presented in Section 6 consists of the rules and axioms of AA plus an axiom and a rule dedicated to the quantifier (Remark 6.2).

It is straightforward to see that the axiomatisation of AAM is as the axiomatisation of AA where only axiom $R_{\top 3}$ needs to be (analogously) reformulated for action models, whereas the axiom R_B is the same in AA and in AAM, except that, clearly, the relation $\rhd_a$ used in that axiom now refers to the much more involved view relation for partial observability defined above, where an agent considers all actions possible that are accessible for her given the actual action. These two relevant axioms are:

$(R'_{\top 3}) \quad \langle \alpha.\mathcal{E}_e \rangle \top \leftrightarrow \langle \alpha \rangle \mathsf{pre}(e);$
$(R'_B) \quad \langle \alpha \rangle \hat{B}_a \varphi \leftrightarrow (\langle \alpha \rangle \top \wedge \bigvee_{\alpha \rhd_a \beta} \hat{B}_a \langle \beta \rangle \varphi).$

Just as for AA we can show that this axiomatisation is complete with respect to the class of models with empty histories, and that this is again a reduction system, such that every formula in the logical language is equivalent to a formula without dynamic modalities $\langle \mathcal{E}_e \rangle$ for action execution and $\langle a \rangle$ for receiving that information.

To prove that this system is a complete axiomatisation of AAM, we need to define a total preorder $\ll$ from a complexity measure $|.|$ which takes into consideration the precondition formulas present in action models $\mathcal{E}_e$. It therefore seems that this demands that

$$\begin{array}{lcl} |(\mathcal{E}, e)| & = & \sum_{e' \in E} |\mathsf{pre}(e')| \\ |\alpha| & = & \sum \{ |(\mathcal{E}, e)| : (\mathcal{E}, e) \text{ occurs in } \alpha \}. \end{array}$$

We wish to investigate this later and thus show completeness.

7.2 Arbitrary Asynchronous Action Model Logic

A further generalisation is the extension of the logical language with a quantifier $\langle\otimes\rangle$ over action models, such that $\langle\otimes\rangle\varphi$ means that φ is true after the execution of some finite action model in the current (s, α) pair of the given model.

Let $\mathcal{AM}_{-\otimes}$ be the class of finite pointed action models where $\langle\otimes\rangle$ does not occur in the preconditions. We then get that

$$w, \alpha \models \langle\otimes\rangle\varphi \text{ iff there exists } \beta \in (\mathcal{AM}_{-\otimes} \cup A)^* \text{ such that } w, \alpha \models \langle\beta\rangle\varphi.$$

Let us call the logic with this quantifier AAAM (an extra A, for Arbitrary). Although work on this logic is also very much work in progress, it is illuminating to compare this extension AAAM of AAM with the logic AAA of this submission, wherein we quantify over histories containing announcements. For the synchronous version of arbitrary action model logic, Hales showed in [10] that the restriction to quantifier-free precondition formulas in action models can be relaxed, and that we can prove the property (not the definition) of this logic that

$$w, \alpha \models \langle\otimes\rangle\varphi \text{ iff there exists } \beta \in (\mathcal{AM} \cup A)^* \text{ such that } w, \alpha \models \langle\beta\rangle\varphi.$$

He also showed that we can *synthesize* a multi-pointed action model $\mathcal{E}_F$ (where $F \subseteq \mathcal{D}(\mathcal{E})$) from φ such that $\langle\otimes\rangle\varphi$ is equivalent to $\langle\mathcal{E}_F\rangle\varphi$.

It it were possible to prove similar results for the logic AAAM of arbitrary asynchronous action models, that would be of great interest, as this would then show that AAAM is as expressive as AAM (without quantification), by reducing every formula to one without quantifiers, unlike the larger expressivity of quantifying over asynchronous announcements in AAA; and it would also show decidability of AAAM. Even independent from that, synthesis of asynchronous partially observable actions, and the complexity of such tasks, seems of interest to investigate further.

8 Conclusion

We presented the logic AAA of arbitrary asynchronous announcements, that can be used to reason about agents receiving and sending each other information under asynchronous conditions. We investigated the properties of the arbitrary announcement quantifier, demonstrated bisimulation invariance, the larger expressivity of the logical language with the quantifier, and we showed preservation after history extension of the fragment of the positive formulas. Then, we provided a complete infinitary axiomatisation. Finally, we tentatively described a further generalisation to quantification over action models.

Appendix

Proof of Prop. 3.1. By $\ll$induction on (α, φ). Trivial for the cases where $(\alpha, \varphi) = (\epsilon, \top)$ and (ϵ, p). For the case where $(\alpha, \varphi) = (\beta.a, \top)$, we note that $w \bowtie \beta.a$ iff $w \bowtie \beta$ and $w, \beta.a \models \top$ iff $w, \beta \models \top$, and thus we can apply induction hypothesis, for $(\beta, \top) \ll (\beta.a, \top)$. For the case $(\alpha, \varphi) = (\beta.\psi, \top)$, we note that $(\beta, \psi) \ll (\beta.\psi, \top)$.

For the cases $(\alpha, \varphi) = (\alpha, \neg\psi)$ and $(\alpha, \psi) = (\alpha, \psi_1 \wedge \psi_2)$, we note that $(\alpha, \psi) \ll (\alpha \neg\psi)$ and $(\alpha, \psi_i) \ll (\alpha, \psi_1 \wedge \psi_2)$.

For the case $(\alpha, \varphi) = (\alpha, \hat{B}_a\psi)$, we have: $w \bowtie \alpha$ iff $w' \bowtie \alpha$ by induction hypothesis applied to $(\alpha, \top)$. If $w, \alpha \models \hat{B}_a\psi$, then there is some $v \in W$ and some history β such that R_awv, $\alpha \triangleright_a \beta$, $v \bowtie \beta$ and $v, \beta \models \psi$. But then there is some $v' \in W'$ with vZv' and $R_aw'v'$ and thus, by induction hypothesis applied to $(\beta, \psi) \ll (\alpha, \hat{B}_a\psi)$, we have $v' \bowtie \beta$, $v', \beta \models \psi$ and thus $w', \alpha \models \hat{B}_a\psi$. The converse is analogous.

For the cases $(\alpha, \psi) = (\alpha, \langle a \rangle\psi)$ and $(\alpha, \psi) = (\alpha, \langle\theta\rangle\psi)$, we note that $(\alpha.a, \psi) \ll (\alpha, \langle a \rangle\psi)$ and $(\alpha.\theta, \psi) \ll (\alpha, \langle\theta\rangle\psi)$.

For the case $(\alpha, \varphi) = (\alpha, \langle ! \rangle\psi)$, we have: on the one hand, $w \bowtie \alpha$ iff $w' \bowtie \alpha$ by induction hypothesis applied to $(\alpha, \top)$. On the other hand, suppose $w, \alpha \models \langle ! \rangle\psi$. Then $w, \alpha \models \langle\beta\rangle\psi$ for some history β which does not contain any occurrences of $\langle ! \rangle$. Therefore $\deg_! \langle\beta\rangle\psi < \deg_! \langle ! \rangle\psi$, and thus by induction hypothesis $w', \alpha \models \langle\beta\rangle\psi$, which entails $w', \alpha \models \langle ! \rangle\psi$. The converse is analogous. □

Proof of Prop. 3.2. It is obvious that condition i. is satisfied. Now, suppose condition ii. fails. That is, for some $v \in W$, we have R_awv but for all (the finitely many) v' such that $R_aw'v'$ it is not the case that vZv'. Let $R'_a[w'] = \{v'_1, ..., v'_n\}$. For each v'_i there exists some pair (α_i, φ_i) such that either $v, \epsilon \models \langle\alpha_i\rangle\varphi_i$ but $v'_i, \epsilon \not\models \langle\alpha_i\rangle\varphi_i$, or $v, \epsilon \not\models \langle\alpha_i\rangle\varphi_i$ but $v'_i, \epsilon \models \langle\alpha_i\rangle\varphi_i$. Let $\theta_i = \langle\alpha_i\rangle\psi_i$ in the former case and $\theta_i = \neg\langle\alpha_i\rangle\psi_i$ in the latter, and call $\psi = \bigwedge_{i=1}^n \theta_i$. Note that $v, \epsilon \models \psi$ and thus $w, \epsilon \models \hat{B}_a\psi$. But then by the definition of Z we have that $w', \epsilon \models \hat{B}_a\psi$, and thus w' has a successor satisfying each formula θ_i: contradiction. Condition iii. is proven similarly. □

Proof of Prop. 3.3. Let $\mathbf{R}_a$ be a relation defined on the set of pairs (s, α) with $s \bowtie \alpha$ as follows:

$$(s, \alpha)\mathbf{R}_a(t, \beta) \quad \text{iff} \quad sR_at, \alpha \triangleright_a \beta, \text{ and } t \bowtie \beta.$$

Note that $s, \alpha \models B_a\varphi$ iff $t, \beta \models \varphi$ for all (t, β) such that $(s, \alpha)\mathbf{R}_a(t, \beta)$. The proof of this result, then consists in showing that $\mathbf{R}_a$ is serial, transitive and Euclidean.

Seriality. Let us see that, for all α, there exists a history β such that $\alpha \triangleright_a \beta$ and $s \bowtie \alpha$ implies $s \bowtie \beta$. Let $n := |\alpha|_a \leq |\alpha|_!$ and let φ be the n-th occurrence of a formula in α, so that $\alpha = \delta.\varphi.\gamma$ for some δ, γ. Let $\beta = \delta.\varphi.a^k$, where k is a natural number such that $|\delta.\varphi|_a + k = n$. Then clearly $\alpha \triangleright_a \beta$, for β contains n times a and exactly the first n formulas of α, and, if $s \bowtie \alpha$, we have that $s \bowtie \delta.\varphi$, because $\delta.\varphi \subseteq \alpha$, and thus $s \bowtie \alpha.\varphi.a^k$. Since R_a is reflexive, this gives that, for any s such that $s \bowtie \alpha$, $(s, \alpha)\mathbf{R}_a(s, \beta)$.

Transitivity. Since R_a and $\triangleright_a$ are both transitive, then, clearly, so is $\mathbf{R}_a$.

Euclidicity. Again, since R_a and $\triangleright_a$ are Euclidean, so is $\mathbf{R}_a$.

Proof of Prop. 3.4. Let model $\mathfrak{M} = (W, R, V)$ and $s \in W$ be given, and let $\alpha \in (\mathcal{L}_{-!} \cup A)^*$ be such that $s, \epsilon \models \langle\alpha\rangle\hat{B}_a\varphi$. Therefore α is a history, $s \bowtie \alpha$ and

$s, \alpha \models \hat{B}_a\varphi$, so that there are t, β such that $R_a st$, $\alpha \rhd_a \beta$, $t \bowtie \beta$, and $t, \beta \models \varphi$. As $t \bowtie \beta$ and $t, \beta \models \varphi$, it follows that $t, \epsilon \models \langle\beta\rangle\varphi$. It therefore follows that $t, \epsilon \models \langle ! \rangle\varphi$. Finally, as $R_a st$, $\epsilon \rhd_a \epsilon$ and $t \bowtie \epsilon$ we conclude $s, \epsilon \models \hat{B}_a\langle ! \rangle\varphi$.

On the other hand, $\hat{B}_a\langle ! \rangle\varphi \rightarrow \langle ! \rangle\hat{B}_a\varphi$ is not ϵ-valid. Consider the model $M = (W, R, V)$ for a single agent a and atom p and where $W = \{s, t\}$, $R_a = W^2$, and $V(p) = \{s\}$. We then have that $s, \epsilon \models \hat{B}_a\langle ! \rangle B_a\neg p$, because $s, \epsilon \models \hat{B}_a\langle \neg p.a\rangle B_a\neg p$ (because $t, \epsilon \models \langle \neg p.a\rangle B_a\neg p$), whereas clearly $s, \epsilon \not\models \langle ! \rangle\hat{B}_a B_a\neg p$. □

Proof of Prop. 4.2. For a single agent we consider the formula $\langle ! \rangle(B_a p \wedge B_a\neg B_a p)$ and proceed as in Prop. 4.1, where in this case we observe that in model N' it holds that $s', (p \vee \neg q).a \models B_a p \wedge B_a\neg B_a p$.

$$N: \begin{tikzcd} t(\neg p) \arrow[loop left, "a"] \arrow[r, leftrightarrow, "a"] & s(p) \arrow[loop right, "a"] \end{tikzcd} \qquad N': \begin{tikzcd} t'(\neg pq) \arrow[r, leftrightarrow, "a"] \arrow[d, leftrightarrow, "a"'] & s'(pq) \arrow[d, leftrightarrow, "a"] \\ u'(\neg p\neg q) \arrow[r, leftrightarrow, "a"] & v'(p\neg q) \end{tikzcd}$$

Proof of Lemma 5.3. We show the following: Let $\varphi \in \mathcal{L}_+$. For all models $M = (W, R, V)$ and $s \in W$, and for all histories α, β with $\alpha \preceq \beta$: if $s \bowtie \alpha$ and $s, \alpha \models \varphi$, then if $s \bowtie \beta$ it holds that $s, \beta \models \varphi$.

The proof is by induction on the structure of (simple positive) φ.

Case $\bot$. If $s, \epsilon \models [\alpha]\bot$, then $s \not\bowtie \alpha$, and thus $s \not\bowtie \beta$, by definition of $\preceq$, which means $s, \epsilon \models [\beta]\bot$.

Case atoms. If $s, \epsilon \models [\alpha]p$, then either $s \not\bowtie \alpha$, in which case $s \not\bowtie \beta$ and thus $s, \epsilon \models [\beta]p$, or $s \bowtie \alpha$ and $s \in V(p)$, in which case $s, \epsilon \models [\beta]p$ as well. The case for $\varphi = \neg p$ is analogous.

Case conjunction. If $s, \epsilon \models [\alpha](\varphi_1 \wedge \varphi_2)$, and assuming $s \bowtie \beta$ (for otherwise it is trivial), we have that $s, \alpha \models \varphi_i$ for $i = 1, 2$ and thus, by induction hypothesis, $s, \beta \models \varphi_i$, whence $s, \epsilon \models [\beta](\varphi_1 \wedge \varphi_2)$.

Case disjunction. Analogous.

Case belief. Suppose $s, \epsilon \not\models [\beta]B_a\varphi$. Then $s, \epsilon \models \langle\beta\rangle\hat{B}_a\neg\varphi$, which means there exist t, δ with $R_a st$, $\beta \rhd_a \delta$ and $t, \delta \not\models \varphi$. By Lemma 5.1, there is a γ with $\alpha \rhd_a \gamma$ and $\gamma \preceq \delta$, which gives, by induction hypothesis, $t, \delta \not\models \varphi$ and thus $s, \alpha \not\models B_a\varphi$.

Case $[!]\varphi$. Suppose $s, \epsilon \not\models [\beta][!]\varphi$. This means that $s, \epsilon \models \langle\beta\rangle\langle ! \rangle\neg\varphi$, i.e. there exists a word $\delta \in (\mathcal{L}_{-!} \cup A)^*$ such that $s \bowtie \beta.\delta$ and $s, \beta.\delta \not\models \varphi$. Since $\alpha \preceq \beta.\delta$, this gives that $s \bowtie \alpha$ and $s, \alpha \not\models \varphi$, and thus $s, \epsilon \not\models [\alpha][!]\varphi$. □

Proof of Lemma 6.3. Checking the first item is easy: if $\varphi \in \mathsf{AAA}$, then $B_a\varphi \in \mathsf{AAA}$ (by necessitation) and $\psi \rightarrow \varphi \in \mathsf{AAA}$ (by classical propositional logic). Therefore $B_a\varphi \in T$ and $\psi \rightarrow \varphi \in T$, and thus $\varphi \in T_{B_a} \cap T_\psi$.

T_{B_a} is closed under modus ponens because if $\varphi \rightarrow \theta \in T_{B_a}$ and $\varphi \in T_{B_a}$, then $B_a(\varphi \rightarrow \theta), B_a\varphi \in T$, which by the K axiom plus modus ponens gives $B_a\theta \in T$ and thus $\theta \in T_{B_a}$. For T_ψ, suppose $\varphi \rightarrow \theta, \varphi \in T_\psi$. Then $\psi \rightarrow (\varphi \rightarrow$

$\theta) \in T$ and $\psi \to \varphi \in T$. But note that the former is logically equivalent to $(\psi \to \varphi) \to (\psi \to \theta)$, and, since T is closed under logical equivalence, this means by modus ponens that $\psi \to \theta \in T$ and thus $\theta \in T_\psi$.

For the third condition, suppose $L([\beta]\varphi) \in T_{B_a}$ for all β. Then $B_a L([\beta]\varphi) \in T$ for all β and, since $B_a L(\#)$ is an admissible form, then $B_a L([!]\psi) \in T$, and thus $L([!]\varphi) \in T_{B_a}$. If $L([\beta]\varphi) \in T_\psi$ for all β, then $\psi \to L([\beta]\varphi) \in T$ for all β and, again, since $\psi \to L(\#)$ is an admissible form, this entails $\psi \to L([!]\varphi) \in T$ and therefore $L([!]\varphi) \in T_\psi$.

With respect to the last statement: $\psi \in T_\psi$ because $\psi \to \psi$ is a tautology; if $\neg\psi \notin T$, then $\psi \to \bot \notin T$ thus $\bot \notin T_\psi$, and if $\varphi \in T$, then (since $\varphi \to (\psi \to \varphi)$ is a tautology) $\psi \to \varphi \in T$ and thus $\varphi \in T_\psi$. □

Proof of Prop. 6.4. Let T_0 be a consistent theory. Let $\{\varphi_k : k \in \omega\}$ be an enumeration of the formulas in $\mathcal{L}$ where each formula appears infinitely many times. For $k \in \omega$ we will construct a consistent theory T_{k+1}, which is a superset of T_k, as follows:

i. If $\neg\varphi_k \notin T_k$, then $T_{k+1} = (T_k)_{\varphi_k}$;

ii. If $\neg\varphi_k \in T_k$ and φ_k is of the form $L([!]\psi)$, then there must exist some $\beta \in (\mathcal{L}_{-!} \cup A)^*$ such that $L([\beta]\psi) \notin T_k$ (for otherwise, by rule iii., we would have that $\varphi_k \in T_k$: contradiction). We set $T_{k+1} = (T_k)_{\neg L([\beta]\psi)}$.

iii. If $\neg\varphi_k \in T_k$ and φ_k is *not* of the form $L([!]\psi)$, then $T_{k+1} = T_k$.

Each T_k is consistent due to the last statement in the previous Lemma. Then $T = \bigcup_{k\in\omega} T_k$ is consistent. T is trivially closed under modus ponens. For any formula φ_k, either $\neg\varphi_k$ was already in the k-th step of the construction, or φ_k was added to T_{k+1}; therefore T cannot have proper consistent supersets closed under modus ponens. Finally suppose $L([\beta]\psi) \in T$ for all β. If $L([!]\psi) \notin T$, then $\neg L([!]\psi) \in T$ and thus $\neg L([!]\psi) \in T_k$ for some k. Let $m > k$ such that $\varphi_m = L([!]\psi)$. By construction there exists a β such that $\neg L([\beta]\psi) \in T_{m+1} \subseteq T$: contradiction. Therefore T is a maximal consistent theory. □

Proof of Prop. 6.6. By induction on (α, φ).

The case $(\alpha, \varphi) = (\epsilon, \top)$ is trivial. The cases $(\alpha, \varphi) = (\alpha'.\psi, \top)$ and $(\alpha'.a, \top)$ follow from the axioms $R_{\top 1}$, $R_{\top_2}$ and $R_{\top 3}$ and the fact that $(\alpha', \psi) \ll (\alpha'.\psi, \top)$ and $(\alpha', \top) \ll (\alpha'.a, \top)$.

The case (α, p) follows from the definition of $V(p)$ and axiom R_p combined with the fact that $(\alpha, \top) \ll (\alpha, p)$.

The cases $(\alpha, \neg\psi)$ and $(\alpha, \psi_1 \vee \psi_2)$ follow from $R_\neg$ and $R_\vee$, respectively, plus the fact that $(\alpha, \psi_i) \ll (\alpha, \psi_1 \vee \psi_2)$ (for the first case), and $(\alpha, \psi) \ll (\alpha, \neg\psi)$ (for the second case).

Let us see the case $(\alpha, \hat{B}_a\varphi)$: if $T, \epsilon \models \langle\alpha\rangle\hat{B}_a\varphi$, then on the one hand we have that $T \bowtie \alpha$ (i.e., $T, \epsilon \models \langle\alpha\rangle\top$, which by induction hypothesis paired with the fact that $(\alpha, \top) \ll (\alpha, \hat{B}_a\psi)$ gives us that $\langle\alpha\rangle\top \in T$), and on the other hand, $S, \beta \models \varphi$ by some S, β such that $R_a TS$, $\alpha \triangleright_a \beta$ and $S \bowtie \beta$. This means that $S, \epsilon \models \langle\beta\rangle\psi$ and thus (by induction hypothesis due to the fact that $(\beta, \psi) \ll (\alpha, \hat{B}_a\psi)$, we have that $\langle\beta\rangle\psi \in S$. This entails that $\hat{B}_a\langle\beta\rangle\psi \in T$

and thus $\langle\alpha\rangle\top \wedge \bigvee_{\alpha \triangleright_a \beta} \hat{B}_a \langle\beta\rangle\psi \in T$, which by R_B gives $\langle\alpha\rangle\hat{B}_a\psi \in T$. For the converse, we use R_B and the Diamond Lemma.

The cases $(\alpha, \langle a\rangle\psi)$ and $(\alpha, \langle\theta\rangle\psi)$ follow directly from the fact that $(\alpha.x, \psi) \ll (\alpha, \langle x\rangle\psi)$ for $x \in \mathcal{L} \cup A$.

Let us see the case $(\alpha, [!]\psi)$. If $T, \epsilon \models \langle\alpha\rangle[!]\psi$, then $T \bowtie \alpha$ and $T, \alpha \models [!]\psi$, which means that, for all $\beta \in (\mathcal{L}_{-!} \cup A)^*$, $T, \epsilon \models \langle\alpha\rangle[\beta]\psi$. By induction hypothesis, noting that $(\alpha, [\beta]\psi) \ll (\alpha, [!]\psi)$ whenever β does not contain occurrences of $[!]$, we have that $\langle\alpha\rangle[\beta]\psi \in T$ for all β and thus $\langle\alpha\rangle[!]\psi \in T$. Conversely, if $\langle\alpha\rangle[!]\psi \in T$, then $\langle\alpha\rangle\top \in T$ (and thus, by IH, $T, \epsilon \models \langle\alpha\rangle\psi$, which means $T \bowtie \alpha$), and, for all $\beta \in (\mathcal{L}_{-!} \cup A)^*$, $\langle\alpha\rangle[\beta]\psi \in T$, which again by induction hypothesis gives $T, \epsilon \models \langle\alpha\rangle[\beta]\psi$ for all β and thus $T, \alpha \models [!]\psi$, whence $T, \epsilon \models \langle\alpha\rangle[!]\psi$. $\square$

References

[1] Balbiani, P., A. Baltag, H. van Ditmarsch, A. Herzig, T. Hoshi and T. D. Lima, *'Knowable' as 'known after an announcement'*, Review of Symbolic Logic **1(3)** (2008), pp. 305–334.

[2] Balbiani, P. and H. van Ditmarsch, *A simple proof of the completeness of APAL*, Studies in Logic **8(1)** (2015), pp. 65–78.

[3] Balbiani, P., H. van Ditmarsch and S. Fernández González, *Asynchronous announcements*, CoRR **abs/1705.03392** (2019).

[4] Balbiani, P., H. van Ditmarsch and S. Fernández González, *From public announcements to asynchronous announcements*, in: *Proc. of 24th ECAI, Santiago*, 2020.

[5] Baltag, A., L. Moss and S. Solecki, *The logic of public announcements, common knowledge, and private suspicions*, in: *Proc. of 7th TARK* (1998), pp. 43–56.

[6] Ben-Zvi, I. and Y. Moses, *Beyond lamport's* Happened-before*: On time bounds and the ordering of events in distributed systems*, Journal of the ACM **61** (2014), pp. 13:1–13:26.

[7] Chandy, K. and J. Misra, *How processes learn*, in: *Proc. of the 4th PODC*, 1985, pp. 204–214.

[8] Degremont, C., B. Löwe and A. Witzel, *The synchronicity of dynamic epistemic logic*, in: *Proc. of 13th TARK* (2011), pp. 145–152.

[9] Genest, B., D. Peled and S. Schewe, *Knowledge = Observation + Memory + Computation*, in: *Proc. of 18th FoSSaCS*, 2015, pp. 215–229, LNCS 9034.

[10] Hales, J., *Arbitrary action model logic and action model synthesis*, in: *Proc. of 28th LICS*, IEEE, 2013, pp. 253–262.

[11] Hales, J., "Quantifying over epistemic updates," Ph.D. thesis, School of Computer Science & Software Engineering, University of Western Australia (2016), https://research-repository.uwa.edu.au/en/publications/quantifying-over-epistemic-updates.

[12] Halpern, J. and Y. Moses, *Knowledge and common knowledge in a distributed environment*, Journal of the ACM **37(3)** (1990), pp. 549–587.

[13] Knight, S., B. Maubert and F. Schwarzentruber, *Reasoning about knowledge and messages in asynchronous multi-agent systems*, Mathematical Structures in Computer Science **29** (2019), pp. 127–168.

[14] Kshemkalyani, A. and M. Singhal, "Distributed Computing: Principles, Algorithms, and Systems," Cambridge University Press, New York, NY, USA, 2008.

[15] Lamport, L., *Time, clocks, and the ordering of events in a distributed system*, Communications of the ACM **21** (1978), pp. 558–565.

[16] Mukund, M. and M. Sohoni, *Keeping track of the latest gossip in a distributed system*, Distributed Computing **10** (1997), pp. 137–148.

[17] Panangaden, P. and K. Taylor, *Concurrent common knowledge: Defining agreement for asynchronous systems*, Distributed Computing **6** (1992), pp. 73–93.

[18] Plaza, J., *Logics of public communications*, in: *Proc. of the 4th ISMIS* (1989), pp. 201–216.

[19] Ramanujam, R., *Local knowledge assertions in a changing world*, in: Y. Shoham, editor, *Proc. of 6th TARK* (1996), pp. 1–14.

[20] van Benthem, J., *One is a lonely number: on the logic of communication*, in: *Logic colloquium 2002. Lecture Notes in Logic, Vol. 27* (2006), pp. 96–129.

[21] van Ditmarsch, H. and B. Kooi, *The secret of my success*, Synthese **151** (2006), pp. 201–232.

[22] van Ditmarsch, H., W. van der Hoek and B. Kooi, "Dynamic Epistemic Logic," Synthese Library **337**, Springer, 2008.